International Journal of Theoretical Physics, Vol. 33, No. 6, 1994

Electromagnetic Field in Two-Dimensional Locally Flat Space-Time

Kingshuk Majumdar[1,2] and Rajkumar Roychoudhury[1,2]

Received December 9, 1992

It is shown that an infinite-dimensional symmetry is present in two-dimensional electromagnetic field theory. The generators of the ensuing Virasoro algebra are explicitly calculated both for periodic and antiperiodic fields.

In two space-time dimensions conformal invariance leads to an infinite set of conserved quantities (Itzykson and Zuber, 1985). In string theory (Green *et al.*, 1988) these are just constraints like $T_{++} = T_{--} = 0$, where T_{++} and T_{--} are the components of the energy-momentum tensor in light-cone notation. In this paper we show that an infinite-dimensional symmetry is present in two-dimensional electromagnetic (or electrostatic, to be precise) field theory which also arises from the constraints of the theory. To draw an analogy with closed strings, we consider a cylindrical two-dimensional space-time and impose a periodic boundary condition on the electromagnetic field A_μ. For the open string, x and t are unbounded $(-\infty < x < \infty, 0 < t < \infty)$. The generators of the ensuing Virasoro algebra are explicitly calculated both for periodic and antiperiodic (twisted) fields.

We start with the general action for a free electromagnetic field in a two-dimensional curved space-time, given by

$$S = \int \mathscr{L}(x)\, d^2x \tag{1}$$

where the Lagrangian density is

$$\mathscr{L}(x) = -\tfrac{1}{4}\sqrt{-g}\, F_{\mu\nu}F^{\mu\nu} \qquad \text{with} \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad (\mu, \nu \equiv 0, 1) \tag{2}$$

[1]Electronics Unit, Indian Statistical Institute, Calcutta 700035, India.
[2]Present address: Physics and Applied Mathematics Unit, Indian Statistical Institute, Calcutta 700035, India.

0020-7748/94/0600-1257$07.00/0

The stress tensor $T^{\mu\nu}(x)$ in general is defined as

$$T_{\mu\nu}(x) = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}(x)} \tag{3}$$

For the electromagnetic field we have

$$T^{\mu\nu}(x) = (g^{\mu\alpha} F_{\alpha\lambda} F^{\lambda\nu} + \tfrac{1}{4} g^{\mu\nu} F_{\alpha\lambda} F^{\alpha\lambda}) \tag{4}$$

Clearly the stress tensor is symmetric, traceless, and gauge invariant.

Being massless, in two dimensions this theory is clearly conformally invariant. The subsequent analysis of the dynamics and quantization of the electromagnetic field are expedited by making a convenient choice of gauge. We choose

$$g_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$$

for the two-dimensional Minkowski metric.

Equation (2) becomes

$$\mathscr{L}(x) = \tfrac{1}{2}(\partial_0 A_1 - \partial_1 A_0)^2 \tag{5}$$

and the corresponding Hamiltonian density is

$$\mathscr{H}(x) = \tfrac{1}{2}[\dot{A}_1^2 - (\partial_1 A_0)^2] \tag{6}$$

Using the Lorentz gauge $\partial_\mu A^\mu = 0$, $\mu = 0, 1$, we obtain the equation of motion as follows:

$$(\partial_0^2 - a_1^2) A^\mu(x, t) = 0 \tag{7}$$

The components of $T^{\mu\nu}$ follows from (4) as

$$\begin{aligned} T^{\infty} &= -T^{11} = \tfrac{1}{2}[\dot{A}_1^2 - (\partial_1 A_0)^2] \\ T^{01} &= T^{10} = 0 \end{aligned} \tag{8}$$

Clearly from (6) we can see that T^{00} is the Hamiltonian density of the system.

Now $T_{10} = 0$ [which follows from (8) after employing the usual procedures for lowering the indices]. Hence if we put $\nu = 0$ in the conservation law $g^{\alpha\mu} \partial_\alpha T_{\mu\nu} = 0$, we obtain $\partial_0 T_{00} = 0$. This implies that there exists an infinite set of conserved quantities.

Case A. The simplest generalization of Minkowski-space quantum field theory is the introduction of nontrivial topological structure in a locally flat space-time. As mentioned before in analogy with the closed string, we consider the $R^1 \otimes S^1$ two-dimensional space-time with compactified spatial sections. This space-time has the two-dimensional

Minkowski-space line element, but the spatial points x and $x+L$ are identified, where L is the periodicity length. This space-time is shown in Fig. 1.

We impose periodic boundary conditions on A's, i.e.,

$$A(x, t) \equiv A(x+nL, t) \tag{9a}$$

One can also consider imposing antiperiodic boundary conditions

$$A(x, t) \equiv (-1)^n A(x+nL, t) \tag{9b}$$

We will discuss the implications of boundary conditions for the field modes later.

Case B. In analogy with the open string, we can also consider the general case

$$-\infty < x < \infty, \qquad 0 < t < \infty \qquad \text{of} \quad \text{space-time}$$

Let us now determine the proper boundary conditions to be used at the boundaries. The usual action is given in equation (1),

$$S = \frac{1}{2}\iint dx\, dt\, (\partial_0 A_1 - \partial_1 A_0)^2 \tag{9c}$$

Demanding the action to be stationary, i.e., $\delta S = 0$, we get the required

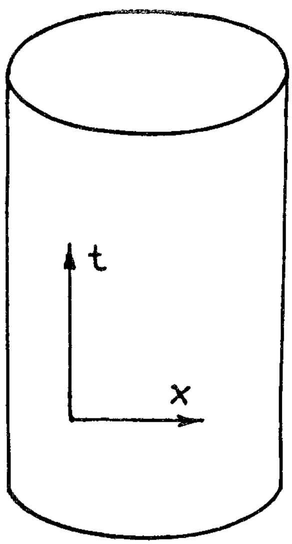

Fig. 1

boundary conditions to be satisfied besides the equation of motion stated earlier:

$$(\partial_0 A_1 - \partial_1 A_0)|_{t=0}^{\infty} = 0 \tag{9d}$$

and

$$(\partial_0 A_1 - \partial_1 A_0)|_{x=0}^{L} = 0 \qquad \text{for case A} \tag{9e}$$

Due to periodicity (or antiperiodicity) in case A, this is automatically satisfied. But for the general case (case B) we have to impose the boundary conditions that

$$(\partial_0 A_1 - \partial_1 A_0)|_{x=\infty} = 0 \tag{9f}$$

$$(\partial_0 A_1 - \partial_1 A_0)|_{x=-\infty} = 0 \tag{9g}$$

separately. Now from the equation of motion we get that $\partial_0 E^1 = 0$, which implies that the Hamiltonian is constrained to be zero, i.e., $H = 0$. The case is similar to string theory. As the electric field E^1 is defined as

$$E^1 = \partial_0 A_1 - \partial_1 A_0 \tag{10}$$

we have

$$E^1(\infty, t) = 0$$

$$E^1(-\infty, t) = 0 \tag{10a}$$

Also, the condition

$$E^1(x, \infty) - E^1(x, 0) = 0 \tag{10b}$$

has to be satisfied for both cases. Equations (10a) and (10b) are free boundary conditions which imply that the energy density at $\pm\infty$ be zero [as energy density E^1 (Green *et al.*, 1988)]. Now the Hamiltonian generates the time evolution of the system. The Fourier components of T_{00} (evaluate $t = 0$) are

$$L_m = \int_0^L e^{2imx} T_{00}\, dx, \qquad \text{for case A} \tag{11}$$

$$L_m = \int_{-\infty}^{\infty} (e^{imx} T_{00} + e^{-imx} T_{11})\, dx \qquad \text{for case B} \tag{12}$$

We restrict ourselves to case A only. Similar results hold for case B. The general expression of $A^\mu(t, x)$ which satisfies equation (7) is

$$A^\mu(t, x) = \sum_{k_1, \lambda} [a_{k_1}^\lambda u_{k_1}^{\mu(\lambda)}(t, x) + a_{k_1}^{\lambda +} u_k^{\mu *(\lambda)}(t, x)] \tag{13}$$

where the plane wave modes $u_k^{\mu(\lambda)}$ are given by

$$u_{k_1}^{(\mu)(\lambda)}(t, x) = \frac{1}{(2Lw)^{1/2}} \varepsilon_{k_1}^{\mu(\lambda)} e^{i(k_1 x - wt)} \tag{14}$$

with

$$\varepsilon_{k_1}^{(\mu)(\lambda)}, \quad \lambda = 0, 1$$

labeling independent polarization vectors associated with mode k_1. From the mass shell condition, $w = |k_1|$. The polarization vectors can be chosen to form an orthonormal system with

$$\eta_{\alpha\beta} \varepsilon_{k_1}^{(\alpha)(\lambda)} \varepsilon_{k_1}^{(\beta)(\lambda')} \equiv \eta_{\lambda\lambda'} \tag{15}$$

The effect of the space closure is to restrict the field modes (14) to a discrete set

$$u_k = (2Lw)^{-1/2} e^{i(k_1 x - wt)} \tag{16}$$

where $k_1 = 2\pi n/L$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. As $w = |k|$, the model labeled by positive values of n have the form $\exp[ik_1(x - t)]$ and represent waves moving left to right, while negative values of n give $\exp[ik_1(x + t)]$, representing left-moving waves.

For the antiperiodic case the modes are given by equation (16) but with $k_1 = 2\pi(n + \frac{1}{2})/L$, $n = 0, \pm 1, \pm 2, \ldots$. In the latter case the electromagnetic field is to be regarded as a section through a nonproduct bundle and we call it a twisted field. Avis and Isham (1979*a,b*) have argued that in most space-times with nontrivial topology, one must include both twisted and untwisted fields. Thus twisted fields should not be considered as a mathematical curiosity, but rather as being equally as important as untwisted fields. Before we proceed further we note that one can choose the polarization vector in two dimensions in the following way: $\varepsilon_\mu^0 = \binom{1}{0}$ and $\varepsilon_\mu^{(1)} = \binom{0}{1}$. We shall now calculate the Fourier mode of the constraint $\dot{A}_1^2 - (\partial_1 A_0)^2$. First we rewrite equation (13) in the following way:

$$A^\mu(t, x) = \sum_{n \geq 0} \sum_\lambda \frac{1}{(2\pi w)^{1/2}} [\varepsilon_n^{\mu\lambda} e^{i(2nx - wt)} a_n^\lambda + \varepsilon_n^{\mu*\lambda} a_n^{\lambda\dagger} e^{-i(2nx - wt)}] \tag{17}$$

for the right-moving waves and a similar expression (with $n < 0$) for the left-moving modes. Also, to draw analogy with the string results, we have taken $L = \pi$. Now we define

$$L_m = \frac{1}{2} \int_0^\pi e^{2imx} [\dot{A}_1^2 - (\partial_1 A_0)^2] \, dx \tag{18}$$

After some straightforward calculations we get

$$L_m = \frac{1}{2} \sum_{n,\lambda} g^{\lambda\lambda} [(n|n-m|)^{1/2} a_n^\lambda a_{n-m}^{\lambda\dagger} + (n|n+m|)^{1/2} a_n^{\lambda\dagger} a_{n+m}^\lambda] \tag{19}$$

Using the commutation relation

$$[a_n^\lambda, a_{n'}^{\dagger\lambda'}] = -\delta_{n,n'} g^{\lambda\lambda'} \tag{20}$$

it can be shown that

$$[L_m, L_n] = (m-n) L_{m+n} \tag{21}$$

which is the famous Virasoro algebra. The steps followed to deduce (20) are valid at the quantum mechanical level for $m+n \neq 0$. For $m+n=0$, the two infinite sums in (19) suffer from normal order ambiguity. To take account of this we modify (2) in the following way:

$$[L_m, L_n] = (m-n) L_{m+n} + A(m) \delta_{m+n} \tag{22}$$

$A(m)$ can be calculated in the customary way. Using the Jacobi identity, namely,

$$[L_k, [L_n, L_m]] + [L_n, [L_m, L_k]] + [L_m, [L_k, L_n]] = 0 \tag{23}$$

it can be shown that $A(m)$ is of the form

$$A(m) = am^3 + bm \tag{24}$$

Now if $|0_L\rangle$ is the vacuum associated with the discrete modes (16), then $|0_L\rangle \to |0\rangle$ as $L \to \infty$, $|0\rangle$ being the usual Minkowski-space vacuum. To calculate $A(m)$ we utilize the fact that $\langle 0|[L_1, L_{-1}]|0\rangle = 0$ and then explicitly calculate the value of $\langle 0|[L_2, L_{-2}]|0\rangle$. It can be shown that a and b are given by

$$a = 1/12, \qquad b = -1/12$$

Hence $A(m)$ can be written as

$$A(m) = \frac{C}{12}(m^3 - m) \tag{25}$$

where $C = 1$ for periodic fields and is equal to 3/4 for antiperiodic fields. We also give the explicit expression for L_0:

$$L_0 = \sum_\lambda g^{\lambda\lambda} \left(\sum_{n=0}^{\infty} n a_n^{\lambda\dagger} a_n^\lambda \right) - \frac{1}{24} \tag{26}$$

and a similar expression for $\tilde{L}_0$; the Hamiltonian is given by

$$H = L_0 + \tilde{L}_0 = \sum_{\lambda} g^{\lambda\lambda}[(\sum n a_n^{\lambda\dagger} a_n^{\lambda}) + (\sum n \tilde{a}_n^{\lambda\dagger} \tilde{a}_n^{\lambda})] - \frac{1}{12} \tag{27}$$

and a similar expression for the antiperiodic fields with n replaced by $n + 1/2$ and the constant terms being $1/24$ instead of $-1/12$.

To conclude, we have found that an infinite-dimensional symmetry can be realized in two-dimensional electromagnetic field theory and the symmetry algebra is the Virasoro algebra. Recently Cappelli *et al.* (1992) have shown that free planar electrons in a uniform magnetic field (in $1+2$ dimensions) possess the W_∞-algebra. This study is of importance for the quantum Hall effect. However, in our case since the study was restricted to $1+1$ dimensions we essentially studied an electrostatic field. An extension of this work in $1+2$ dimensions is likely to yield richer algebraic structure.

REFERENCES

Avis, S. J., and Isham, C. J. (1979*a*). *Nuclear Physics B*, **156**, 441.

Avis, S. J., and Isham, C. J. (1979*b*). Quantum field theory and fiber bundles in a general space-time, in *Recent Developments in Gravitation, Cargèse 1978*, M. Levy and S. Deser, eds., Plenum Press, New York.

Birrel, N. D., and Davis, P. C. W. (1982). *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge.

Cappelli, A., Trugenberger, C. A., and Zemba, G. R. (1992). CERN-TH 6516/92.

Isham, C. J. (1978). *Proceedings of the Royal Society of London A*, **362**, 383.

Green, M. B., Schwartz, J. H., and Witten, E. (1988). *Superstring Theory*, Cambridge University Press, Cambridge, Vol. I.

Itzykson, C., and Zuber, J. B. (1985). *Quantum Field Theory*, McGraw-Hill, New York.

Kaku, M. (1988). *Introduction to Superstrings*, Springer-Verlag.